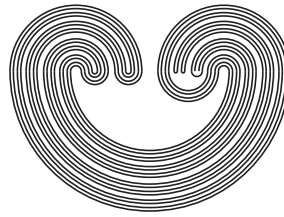


<http://topology.auburn.edu/tp/>

TOPOLOGY PROCEEDINGS



Volume 59, 2022

Pages 25–50

<http://topology.nipissingu.ca/tp/>

A NOTE ON η_1 -SPACES

by

HAROLD BENNETT, SHELDON DAVIS AND DAVID LUTZER

Electronically published on November 17, 2020

This file contains only the first page of the paper. The full version of the paper is available to Topology Proceedings subscribers. See <http://topology.auburn.edu/tp/subscriptioninfo.html> for information.

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: (Online) 2331-1290, (Print) 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

A NOTE ON η_1 -SPACES

HAROLD BENNETT, SHELDON DAVIS, AND DAVID LUTZER

ABSTRACT. In this paper we study some topological properties of η_1 -spaces, i.e., topological spaces that use the open-interval topology of the η_1 -sets that were introduced by Hausdorff more than a century ago. We focus on paracompactness, normality of products, topological completeness of various kinds, and certain generalized metric properties such as the existence of a small diagonal. In many cases, we find an intimate relation between topological properties of small η_1 -spaces (i.e. having cardinality 2^ω) and the Continuum Hypothesis (CH). For example, we show that (CH) is equivalent to the statement that if X is an η_1 -space of cardinality 2^ω , then X^n is hereditarily paracompact and monotonically normal and is homeomorphic to X for every finite $n \geq 1$, and we show that CH is equivalent to the statement that every η_1 -space of cardinality 2^ω is realcompact. In addition, we investigate the role of Hušek's small diagonal property, showing that an η_1 -space X has a small diagonal if and only if each subset $S \subseteq X$ with $|S| \leq \omega_1$ is closed. Consequently, under CH, no η_1 -space with cardinality 2^ω can have a small diagonal, and we show that that if CH fails, then it is undecidable whether each ultrapower $\mathbb{R}^\omega/\mathcal{U}$ must have a small diagonal. Under CH, we show that any finite power of any GO-modification of a small η_1 -set is both monotonically normal and paracompact, and is homeomorphic to its square. We pose several questions about the topology of small η_1 -spaces in models where the Continuum Hypothesis fails.

In memoriam: We dedicate this paper to the memory of our friend and colleague Phillip Zenor whose work has deeply influenced our research.

2020 *Mathematics Subject Classification.* Primary 54F05; Secondary 54G15, 54G52, 54D20.

Key words and phrases. η_1 -set, η_1 -space, paracompact, monotonically normal, Dieudonné complete, realcompact, A-metric space, Baire space, $C_p(X)$, small diagonal, ultrapower, homeomorphic to square, Continuum Hypothesis.

©2020 Topology Proceedings.

This file contains only the first page of the paper. The full version of the paper is available to Topology Proceedings subscribers. See <http://topology.auburn.edu/tp/subscriptioninfo.html> for information.