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ADDING A CONTINUOUS MAP BY FORCING

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Abstract. We study how forcing adds continuous maps. We prove the following theorems. Let X be a completely regular space.

(1) If X is a scattered compact Hausdorff space and Y is a discrete space, then forcing does not add any continuous maps from X to Y.

(2) If X is not a zero-dimensional scattered pseudocompact space and Y is a topological space which has more than one point, then some ccc forcing adds a continuous map from X to Y.

(3) If X is infinite and there is a homeomorphism from X onto X which is not the identity map, then some ccc forcing adds a homeomorphism from X onto X.

1. Introduction

Let $\mathbf{V}$ be a ground model. For a forcing $\mathbb{P}$, let $\mathbf{V}^{\mathbb{P}}$ denote the forcing extension of $\mathbf{V}$ by $\mathbb{P}$. Let $(X, \mathcal{T}_X)$ be a topological space in $\mathbf{V}$. In $\mathbf{V}^{\mathbb{P}}$, we define a topological space $(X, \mathcal{T}_X^{\mathbb{P}})$ such that $\mathcal{T}_X^{\mathbb{P}}$ is the topology generated by $\mathcal{T}_X$ in $\mathbf{V}^{\mathbb{P}}$. That is, $\mathcal{T}_X^{\mathbb{P}} = \{\bigcup \mathcal{U} : \mathcal{U} \subseteq \mathcal{T}_X\}$. In general, we have $\mathcal{T}_X \subsetneq \mathcal{T}_X^{\mathbb{P}}$ because forcing $\mathbb{P}$ may add a new open set.

Forcing adds new sets. Cohen forcing adds a Cohen real. Hechler forcing adds a function which dominates the functions from ω to ω in the ground model. Forcing with a Souslin tree adds an uncountable antichain of the tree. In [5], we studied how forcing adds convergent sequences.

Forcing preserves continuity of a map; that is, if $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a continuous map, then $f : (X, \mathcal{T}_X^{\mathbb{P}}) \to (Y, \mathcal{T}_Y^{\mathbb{P}})$ remains continuous for

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