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by

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# A SURVEY OF BOUNDS ON THE CARDINALITY OF A HAUSDORFF SPACE

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Abstract. Improving bounds for the cardinality of a Hausdorff space has been an active area of research in set-theoretic topology for roughly a century. These bounds are expressed in terms of cardinal functions; that is, functions that "measure" a property of a topological space using infinite cardinals. In this survey we study the many results on this topic, give new bounds, and reframe certain proofs. The theorems of Arhangel′skiĭ, Hajnal-Juhász,
De Groot-Smirnov, and Pospišil are reviewed, including their many subsequent improvements. Bounds involving variations of the extent and the diagonal degree are given, along with several "double exponent" bounds. Proofs of the various improvements of Pospišil's theorem are reorganized with a new unified approach. Certain full proofs are given to illustrate techniques involving one-to-one maps, closing-off arguments, the Erdős-Rado theorem, and elementary submodels. New results include $|X| \leq 2^{wL_c(X)\Delta_2(X)}$, $|X| \leq 2^{qaL_c(X)wt(X)\psi_c(X)}$, $|X| \leq 2^{nw_\theta(X)}$, and $|X| \leq 2^{e_\theta(X)s\Delta(X)}$ if $X$ is Hausdorff.

## 1. Introduction

In the early twentieth century it was shown that the cardinality of a second countable Hausdorff space is at most $\mathfrak{c}$, the cardinality of the real line. This implies, for example, that the cardinality of a compact metric space is at most $\mathfrak{c}$. (In fact, Lindelöf will suffice in place of compact).

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