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CACTUS GROUPS FROM THE VIEWPOINT OF GEOMETRIC GROUP THEORY

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Abstract. Cactus groups and their pure subgroups appear in various fields of mathematics and are currently attracting attention from diverse mathematical communities. They share similarities with both right-angled Coxeter groups and braid groups. In this article, our goal is to highlight the tools offered by geometric group theory for the group theoretical study of these groups. Among the new contributions made possible thanks to this geometric perspective, we describe an explicit and efficient solution to the conjugacy problem, and we prove that cactus groups are virtually cocompact special and acylindrically hyperbolic. This has various algebraic consequences. From a purely geometrical point of view, we also prove that cactus groups are pairwise non-quasi-isometric.

1. Introduction

Cactus groups J_n first appeared in [11] and [9], under the names "quasibraid groups" and "mock reflection groups," respectively. The group J_n is the fundamental group of the quotient orbifold of $\overline{M}_0^{n+1}(\mathbb{R})$, the Deligne–Knudson–Mumford moduli space of stable real curves of genus 0 with $n+1$ marked points, by the action of the symmetric group S_n that permutes the first n of those points. The picture of stable real curves in this space, which looks like a cactus, gave the name "cactus groups" [25].

Given an integer $n \geq 2$, the *cactus group* J_n can be defined by the generators $s_{p,q}$, with $1 \leq p < q \leq n$, submitted to the following relations:

- $s_{p,q}^2 = 1$ for every $1 \leq p < q \leq n$;

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