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by

Tonatiuh Matos-Wiederhold and Luciano Salvetti

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Web:	http://topology.nipissingu.ca/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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UNCOUNTABLE SETS AND AN INFINITE LINEAR ORDER GAME

TONATIUH MATOS-WIEDERHOLD AND LUCIANO SALVETTI

Abstract. An infinite game on the set of real numbers appeared in Matthew Baker's work [Math. Mag. 80 (2007), no. 5, pp. 377–380] in which he asks whether it can help characterize countable subsets of the reals. This question is in a similar spirit to how the Banach-Mazur Game characterizes meager sets in an arbitrary topological space.

In a recent paper, Will Brian and Steven Clontz prove that in Baker's game, Player II has a winning strategy if and only if the payoff set is countable. They also asked if it is possible, in general linear orders, for Player II to have a winning strategy on some uncountable set.

To this we give a positive answer and moreover construct, for every infinite cardinal κ, a dense linear order of size κ on which Player II has a winning strategy on *all* payoff sets. We finish with some future research questions, further underlining the difficulty in generalizing the characterization of Brian and Clontz to linear orders.

1. Introduction

In [2], Matt Baker introduces the following game, called the *Cantor Game*, on the real numbers: Fix a subset $S \subseteq \mathbb{R}$. Player I starts by picking a real number a_0. Then, Player II picks a real number b_0 such that $a_0 < b_0$. In the n-th turn, Player I picks a real number a_n such that $a_{n-1} < a_n < b_{n-1}$ and then Player II picks a real number b_n such that $a_n < b_n < b_{n-1}$. After ω-many turns they form two sequences of real numbers $\{a_n\}$ and $\{b_n\}$ such that $a_0 < a_1 < \cdots < b_1 < b_0$. We say that

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