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by

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GRUENHAGE SPACES AND THEIR INFLUENCE ON BANACH SPACE RENORMING THEORY

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This article is dedicated to the memory of Prof. Gary F. Gruenhage (1947 – 2023).

ABSTRACT. In a paper from 1987, Gruenhage defined a class of topological spaces that now bear his name, and used it to solve a problem of Talagrand on the existence of dense G_δ metrizable subsets of Gul'ko compact spaces. Gruenhage's paper became highly influential among researchers in renorming theory, a branch of Banach space theory. In this paper we survey Gruenhage and related spaces, and their interactions with renorming theory.

1. INTRODUCTION

Let X be a topological space. In this paper we shall consider only Hausdorff spaces. We call a family $\mathscr{U}$ of subsets of X T_0*-separating* or simply *separating* if, given distinct $x, y \in X$, there exists $U \in \mathscr{U}$ such that $\{x, y\} \cap U$ is a singleton (so in general it is not possible to choose U so that $\{x, y\} \cap U = \{x\}$). Given a family $\mathscr{U}$ of subsets of X and $x \in X$, we write

$$\operatorname{ord}(x, \mathscr{U}) = \operatorname{card}\{U \in \mathscr{U} \ : \ x \in U\}.$$

In his brief yet influential 1987 study of Gul'ko compact spaces, Gruenhage defined the following concept.

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