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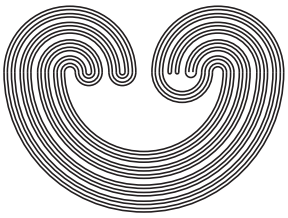

A new ultrafilter proof of Van der Waerden's Theorem

by

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A NEW ULTRAFILTER PROOF OF VAN DER WAERDEN'S THEOREM

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Abstract. We present a new short proof of Van der Waerden's Theorem about the existence of arbitrarily long monochromatic arithmetic progressions. The proof uses algebra in the compact space of ultrafilters $\beta\mathbb{N}$, but contrarily to the other existing proofs, neither minimal nor idempotent ultrafilters are involved.

Introduction

Van der Waerden's Theorem is one of the fundamental results of combinatorics, with a large number of applications across mathematics. It states that for every finite partition (coloring) $\mathbb{N} = C_1 \cup \ldots \cup C_r$ there exist arbitrarily long "monochromatic" arithmetic progressions $a, a+d, \ldots, a+(\ell-1)d \in C_i$. After the original purely combinatorial proof based on an ingenious and rather complex double inductive procedure [10], several others have been found over the years (for example, a remarkable proof was found by S. Shelah in [9]).

An ultrafilter proof was first found in 1989 by V. Bergelson, H. Furstenberg, N. Hindman, and Y. Katznelson [1] using minimal idempotent ultrafilters in the right topological compact semigroup $(\beta\mathbb{N}, \oplus)$; in fact, every set that belongs to a minimal ultrafilter contains arbitrarily long arithmetic progressions, as was proven shortly thereafter by V. Bergelson and N. Hindman [2]. About fifteen years later, in 2004, S. Koppelberg

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