On the reversibility of topological spaces and related notions.

Vitalij A.Chatyrko

May 25, 2018

Content

- Introduction.
- (Non)reversible topological spaces.
- The reversibility of separable metric connected manifolds.
- A generalization of the reversibility to categories.
  g-reversible topological groups.
- Some references

Introduction, I

Proposition 1.

Let U, V be topological spaces such that U is compact and V is Hausdorff, and $f : U \to V$ be a continuous bijection. Then f is a homeomorphism.

Corollary 1.

Let X be a compact Hausdorff topological space. Then every continuous bijection of X onto itself is a homeomorphism.

Corollary 2.

Let Y be a set and τ be a topology on Y such that the topological space (Y, τ) is compact Hausdorff. Then there exists no topology σ on Y such that the topological spaces (Y, τ) and (Y, σ) are homeomorphic and σ is a proper subset of τ (or τ is a proper subset of σ).

Introduction, II

Question 1

For what topological spaces X is every continuous bijection of X onto itself a homeomorphism?

Question 2

For what set Y and a topology τ on Y does there exist a topology σ on Y such that σ is a proper subset of τ, and the topological spaces (Y, τ) and (Y, σ) are homeomorphic?

Question 3

For what set Y and a topology τ on Y does there exist a topology σ on Y such that τ is a proper subset of σ, and the topological spaces (Y, τ) and (Y, σ) are homeomorphic?

Introduction, III

Proposition 2

Let Y be a set and τ be a topology on Y. Then the following statements are equivalent:

(a) every continuous bijection of the topological space (Y, τ) onto itself is a homeomorphism;

(b) there exists no topology σ on Y such that σ is a proper subset of τ, and the topological spaces (Y, τ) and (Y, σ) are homeomorphic;

(c) there exists no topology σ on Y such that τ is a proper subset of σ, and the topological spaces (Y, τ) and (Y, σ) are homeomorphic.

Reversible topological spaces, I

Definition 1 (Rajagopalan and Wilansky, 1966)

A topological space X is called *reversible* if each continuous bijection of X onto itself is a homeomorphism.

Trivial examples of reversible spaces are compact Hausdorff spaces, discrete spaces D_τ of cardinality τ, topological spaces with finite topologies, in particular, finite spaces.

A topological space X is called *locally* R^n if for each point $x \in X$ there is an open nbd Ox of x which is homeomorphic to the n-dimensional Euclidean space R^n for some positive integer n, the same for all points of X.

Proposition 3 (RW, 1966)

Each locally R^n topological space is reversible. In particular, any connected n-dimensional manifolds wihout a boundary and any space of the form $D_\tau \times R^n$ are reversible.

(Non)reversible topological spaces, I

Question 4 (RW, 1966)

What spaces are (non)reversible?

Proposition 4 (RW, 1966)

Topological unions of finitely many connected reversible spaces are reversible.

Proposition 5 (RW, 1966)

Let X be a non-reversible topological space and Y any topological space. Then the topological union $X \oplus Y$ (respectively, topological product $X \times Y$) is not reversible.

Nonreversible topological spaces, I

Proposition 6

Let X, Y be topological spaces and $p \in X$. Let also $X \setminus \{p\}$ be a disjoint union $\cup\{X_i : i = \pm 1, \dots\}$ of open subsets of X such that each X_i is homeomorphic to Y and the family of sets $\{p\} \cup (\cup_{i \leq k} X_i)$, where $k \leq -1$, is a base of X at p. Then the space X is not reversible.

Corollary 3 (RW, 1966)

The space $\mathbb{Q}$ (respectively, $\mathbb{P}$) of rational (respectively, irrational) numbers is nonreversible.

Obs $\mathbb{R} = \mathbb{Q} \cup \mathbb{P}$ is reversible.

Nonreversible topological spaces, II

Denote the subspace of the real line consisting of the positive integers, 0 and the reciprocals of the positive integers by $N_{\aleph_0}$.

Note that $N_{\aleph_0}$ is homeomorphic to the topological sum $D_{\aleph_0} \oplus cD_{\aleph_0}$, where $cD_{\aleph_0}$ is the one-point compactification of $D_{\aleph_0}$, and by Proposition 6 it is nonreversible.

Obs Both spaces $D_{\aleph_0}, cD_{\aleph_0}$ are reversible.

Corollary 4 (Chatyrko and Hattori, 2016)

The Sorgenfrey line is nonreversible.

But the Khalimski line (and its finite powers) is reversible.

Nonreversible topological spaces, III

Proposition 7 (RW, 1966)

No infinite dimensional normed space is reversible. In particular, l_2 is not reversible.

Proposition 8 (RW, 1966)

There is a locally compact abelian group which is not reversible.

Some exotic reversible topological spaces, I

In 2010 Hattori defined a natural family $\mathcal{H} = \{\tau(A) : A \subseteq \mathbb{R}\}$ of topologies on the reals indexed by the subsets of $\mathbb{R}$ such that $\tau(\emptyset)$ is the Sorgenfrey topology, $\tau(\mathbb{R})$ is the Euclidean topology and if $A \subseteq B$ then $\tau(B) \subseteq \tau(A)$.

A space is called *super rigid* if it has no continuous self-bijections other than the identity. Each super rigid space is reversible.

Proposition 9 (Kulesza, 2017)

There are $2^{\mathfrak{c}}$ topologically distinct super rigid Hattori spaces.

Hereditarily reversible topological spaces, I

Definition 2

A topological space is called *hereditarily reversible* if each its subspace is reversible.

Discrete spaces D_κ, $\kappa \geq 1$, and one-point compactifications cD_κ of $D_\kappa, \kappa \geq \aleph_0$, are simple examples of hereditarily reversible spaces.

Let X be a space and $p \in X$. We denote by $\chi(p, X)$ the character of X at the point p and by $\chi(X)$ the character of X.

Proposition 10 (Chatyrko, Han, Hattori, 2017)

Let X be a infinite Hausdorff space and $\chi(X) \leq \aleph_0$. If X is neither homeomorphic to D_κ for any $\kappa \geq \aleph_0$ nor $cD_{\aleph_0}$ then X contains a copy of $N_{\aleph_0}$ and hence X is not hereditarily reversible.

Hereditarily reversible topological spaces, II

Corollary 5 (ChHH, 2017)

The only hereditarily reversible Hausdorff spaces X with $\chi(X) \leq \aleph_0$ are spaces homeomorphic to D_κ for some $\kappa \geq 1$ or $cD_{\aleph_0}$.

Proposition 11 (ChHH, 2017)

There exist $2^{\mathfrak{c}}$ countable hereditarily reversible Tychonoff spaces which are pairwise non-homeomorphic.

The reversibility of separable metric connected manifolds, I

All manifolds considered here are separable metric connected.

Recall that compact manifolds and manifolds without boundary are reversible. Note that each of the four different 1-dimensional manifolds $S^1, [0, 1], [0, 1), (0, 1)$ is reversible.

Proposition 12 (Doyle and Hocking, 1976)

For each $n \geq 2$ there exist nonreversible n-manifolds.

Proposition 13 (DH, 1976)

Let M be a manifold. If BdM is compact then M is reversible.

Proposition 14 (DH, 1976)

Let M be an n-dimensional manifold that embeds in $\mathbb{R}^n$. Then if each boundary component of M is compact then M is reversible.

The reversibility of separable metric connected manifolds, II

Note that the products $\mathbb{D}_\tau \times [0,1)$ and $\mathbb{D}_\tau \times \mathbb{C}$, where $\tau \geq \aleph_0$ and $\mathbb{C}$ is the Cantor set, are not reversible.

It was natural to ask if the topological product of two connected reversible spaces is reversible. (Recall that the topological union of two connected reversible spaces is reversible.)

Proposition 15 (Chatyrko and Karassev, 2017)

There exists a 2-dimensional connected manifold M without a boundary such that $M \times [0,1]$ is not reversible.

A generalization of the reversibility to categories, I.

This part is a joint presentation with D. Shakhmatov.

Consider a category $\mathcal{K}$ consisting of a class $ob(\mathcal{K})$ of objects and a class $hom(\mathcal{K})$ of morphisms.
Recall that a morphism $f \in hom(X, Y)$ is called

(a) *monomorphism* if $fg_1 = fg_2 => g_1 = g_2$ for all $g_i \in hom(\cdot, X)$,

(b) *epimorphism* if $g_1 f = g_2 f => g_1 = g_2$ for all $g_i \in hom(Y, \cdot)$,

(c) *bimorphism* if it is both a monomorphism and an epimorphism,

(d) *isomorphism* if there is a morhism $g \in hom(Y, X)$ such that $fg = 1_Y$ and $gf = 1_X$, where 1_X and 1_Y are identity morphisms.

Definition 3

An object $X \in ob(\mathcal{K})$ is called $\mathcal{K}$*-reversible* if each morphism $f \in hom(X, X)$, which is also a bimorphism, is isomorphism.

A generalization of the reversibility to categories, II.

Proposition 16

Let $\mathcal{K}_1$ be the category of topological spaces and continuous mappings. Then a topological space X is $\mathcal{K}_1$-reversible iff it is reversible.

Let $(G_1, \tau_1), (G_2, \tau_2)$ be topological groups. An algebraic isomorphism $h : G_1 \to G_2$ is called *a continuous isomorphism* of (G_1, τ_1) onto (G_2, τ_2) if h is continuous with respect to the topologies τ_1 and τ_2. Moreover, if h is a homeomorphism then h is called *a topological isomorphism* of the topological groups and the groups themselves in the case are called *topologically isomorphic*.

Proposition 17

Let $\mathcal{K}_2$ be the category of topological groups and continuous homomorphisms. Then a topological group G is $\mathcal{K}_2$-reversible iff each continuous isomorphism of G onto itself is a topological isomorphism.

g-reversible topological groups, I.

We will call $\mathcal{K}_2$-reversible topological groups *g-reversible*.

Proposition 18

Let (G, τ) be a topological group, where G is a group and τ a topology on G. Then the following statements are equivalent:

(a) every continuous isomorphism of (G, τ) onto itself is a topological isomorphism;

(b) there exists no topology σ on G such that σ is a proper subset of τ, (G, σ) is a topological group and the topological groups (G, τ) and (G, σ) are topologically isomorphic;

(c) there exists no topology σ on G such that τ is a proper subset of σ, (G, σ) is a topological group and the topological groups (G, τ) and (G, σ) are topologically isomorphic.

g-reversible topological groups, II.

Proposition 19

Each reversible topological group is g-reversible. In particular, discrete groups, compact Hausdorff groups, the topological products of type $G \times R^n$, where G is a discrete group and R^n is the additive group of n-dimensional vectors with the Euclidean topology, are g-reversible.

We are interested in

(A) examples of topological groups which are g-reversible but not reversible as topological spaces, and

(B) examples of groups which are not g-reversible.

g-reversible topological groups, III.

Proposition 20

Each dense subgroup of $\mathbb{R}^n$ is g-reversible.

Corollary 6

Any topological group $\mathbb{Q}^m \times \mathbb{R}^n$, where $m \geq 1, n \geq 0$, and any its dense subgroup are g-reversible. (Note that the group $\mathbb{Q}^m \times \mathbb{R}^n$ is non-reversible as a topological space.)

Corollary 7

Every subgroup G of $\mathbb{R}$ is g-reversible (because it is either dense or discrete). In particular, each subgroup of $\mathbb{R}$ is hereditarily g-reversible.

g-reversible topological groups, IV.

Question 5

Do there exist an integer $n \geq 2$ and a subgroup G of $\mathbb{R}^n$ such that G is non-g-reversible?

Proposition 21

Every Polish topological group is g-reversible. In particular, each closed subgroup of $\mathbb{R}^n$ is g-reversible (in fact it is reversible as a topological space).

Corollary 8

The topological vector space l_2, the topological groups $\mathbb{Z}^{\aleph_0}$ and $\mathbb{R}^{\aleph_0}$ are g-reversible. None of them is reversible.

Proposition 22

Every σ-compact locally compact group is g-reversible.

Corollary 9

Every closed subgroup H of a σ-compact locally compact group G is g-reversible.

Corollary 10

Let $G_i, i = 1, 2$ be σ-compact locally compact topological groups. Then the topological product $G_1 \times G_2$ is g-reversible. In particular,

(a) any product $\mathbb{Z}^m \times \{0, 1\}^\lambda \times \mathbb{R}^n$, where λ is an infinite cardinal and m, n are nonnegative integers, is g-reversible.

(b) Let G_1 be a countable group with the discrete topology and G_2 a compact group. Then $G_1 \times G_2$ is g-reversible.

g-reversible topological groups, VI.

Let G be an infinite abelian group.
A character of G is any homomorphism $\chi : G \to \mathbb{S}$, where $\mathbb{S}$ is a subgroup of the group of complex numbers defined by $|z| = 1$.
Bohr topology on G is the coarsest topology on G that makes continuous all the characters of G.

Proposition 23

Every infinite abelian group with the Bohr topology is g-reversible.

Non-g-reversible topological groups, I.

Proposition 24

For every infinite compact group K, there exists a locally compact group topology on the group $G = K^{\aleph_0}$ which makes it into a non-g-reversible topological group.

Remark 1

By taking $K = \{0, 1\}^{\aleph_0}$ in the statement above, we get a *locally compact abelian group G of order* 2 *which is not g-reversible*. Since G is homeomorphic to the topological product $D_{\mathfrak{c}} \times \mathbb{C}$, where $\mathbb{C}$ is the Cantor set and $\mathfrak{c}$ is the cardinality of continuum, the topological group G is additionally zero-dimensional and metrizable.

Proposition 25

Let G_1 be a non-g-reversible topological group and G_2 be any topological group. Then the product $G_1 \times G_2$ is not g-reversible.

Non-g-reversible topological groups, II.

Another way to produce non-g-reversible topological groups is the following. Recall that for each Tychonoff space X there exists a free abelian precompact topological group $AP(X)$.

Proposition 26

Let X be a space and let $AP(X)$ be the free abelian precompact group of X. If X is not reversible, then $AP(X)$ is not g-reversible.

Corollary 11

There exists a countable precompact metric abelian group which is not g-reversible.

Some references

V. A. Chatyrko, S-E. Han and Y. Hattori, On hereditarily reversible spaces, Topology Appl. 225 (2017) 53-66

V. A. Chatyrko and Y. Hattori, On reversible and bijectively related topological spaces, Topology Appl. 201 (2016) 432-440

V. A. Chatyrko and A. Karassev, Reversible spaces and products, Topology Proc. 49 (2017) 317-320

V. A. Chatyrko and D. Shakhmatov, g-reversible topological groups, manuscript, 2017

P. H. Doyle and J. G. Hocking, Continuous bijections on manifolds, J. Austral. Math.Soc. 22 (Series A) (1976), 257-263.

J. S. Kulesza, Results on spaces between the Sorgenfrey and usual topologies on R. Topology Appl. 231 (2017), 266-275.

M. Rajagopalan and A. Wilansky, Reversible topological spaces, J. Austral. Math. Soc. 61 (1966), 129-138.